EXERCISES

4.1 Prove that isomorphism of field extensions is an equivalence relation.

15.8 4.2 Find the subfields of $\mathbb{C}$ generated by:

(a) $\{0,1\}$
(b) $\{0\}$
(c) $\{0,1,i\}$
(d) $\{i,\sqrt{2}\}$
(e) $\{\sqrt{2},\sqrt{3}\}$
(f) $\mathbb{R}$
(g) $\mathbb{R}\cup\{i\}$

4.3 Describe the subfields of $\mathbb{C}$ of the form

(a) $\mathbb{Q}(\sqrt{2})$
(b) $\mathbb{Q}(i)$
(c) $\mathbb{Q}(\alpha)$ where α is the real cube root of 2
(d) $\mathbb{Q}(\sqrt{5},\sqrt{7})$
(e) $\mathbb{Q}(i\sqrt{11})$
(f) $\mathbb{Q}(\mathrm{e}^2+1)$
(g) $\mathbb{Q}(\sqrt[3]{\pi})$

4.4 This exercise illustrates a technique that we will tacitly assume in several subsequent exercises and examples.

Prove that $1, \sqrt{2}, \sqrt{3}, \sqrt{6}$ are linearly independent over $\mathbb{Q}$.

(*Hint:* Suppose that $p+q\sqrt{2}+r\sqrt{3}+s\sqrt{6}=0$ with $p,q,r,s\in\mathbb{Q}$. We may suppose that $r\neq 0$ or $s\neq 0$ (why?). If so, then we can write $\sqrt{3}$ in the form

$$\sqrt{3}=\frac{a+b\sqrt{2}}{c+d\sqrt{2}}=e+f\sqrt{2}$$

where $a,b,c,d,e,f\in\mathbb{Q}$. Square both sides and obtain a contradiction.)

4.5 Show that $\mathbb{R}$ is not a simple extension of $\mathbb{Q}$ as follows:

(a) $\mathbb{Q}$ is countable.
(b) Any simple extension of a countable field is countable.
(c) $\mathbb{R}$ is not countable.

4.6 Find a formula for the inverse of $p+qi+r\sqrt{5}+si\sqrt{5}$, where $p,q,r,s\in\mathbb{Q}$.